1

Basics of LC Oscillators

1 Introduction

Phase noise has been one of the most interesting yet poorly understood topics in circuit design. The challenge of predicting the amount of phase noise in a given circuit has been approached from a variety of angles, from using a "heuristic model without formal proof" [1] to simulation techniques that lead to accurate results [2][3]. Completing this book involved hours of computer simulation, pages of algebra, translation of antiquated notation, and utilizing archeological-type efforts to unearth significant but otherwise forgotten papers.

With regard to this work, three major steps lead to the solution of this problem. The first is Lesson's equation. All works on phase noise must reference Lesson's equation because it is simple, intuitive, and has withstood the test of time [1]. The second innovation was in the development of Cadence, Inc.'s SpectreRF simulation tool [2]. This tool accurately predicts phase noise and served as a test bench to validate all derived equations. Finally, Huang [4] showed it was possible to write out equations for phase noise explicitly. This work takes inspiration and elements from all these works and forms a model that is as intuitive as Leeson, as accurate as SpectreRF, and as rigorous as Huang.

2 The Mathematical Oscillator

The mathematical model of an ideal voltage-controlled oscillator starting at $t = 0$ is described by the following expression:

$$v(t) = A(t)\sin\left(\omega_0 t + K_v \int_0^t v_c(u)\,du\right),\qquad(1)$$

where v_c is the control voltage of the oscillator as a function of time. The center frequency of oscillation is ω_0 and the instantaneous frequency of oscillation is given by:

$$\omega(t) = \omega_0 + K_v v_c(t),\qquad(2)$$

where K_v is the oscillator sensitivity and typically given in rad/sec/volt.

As can be seen from (1), the small signal model of an oscillator in frequency/ phase domain, with voltage as an input and phase as an output, functions as an integrator. The frequency of oscillation is directly proportional to the control voltage and the oscillator phase is the time-integral of frequency. Being a self-timed system, the oscillator lacks the ability to correct for its own phase. Imagine an oscillator running at some frequency with a constant bias applied to its control voltage line. Any disturbance on the control line will result in instantaneous frequency shift that integrates over the time the disturbance lasts. The resulting phase error will last indefinitely and can never be recovered even though the disturbance lasted for a short amount of time as shown in Figure 1.

FIGURE 1 *Phase jitter accumulation.*
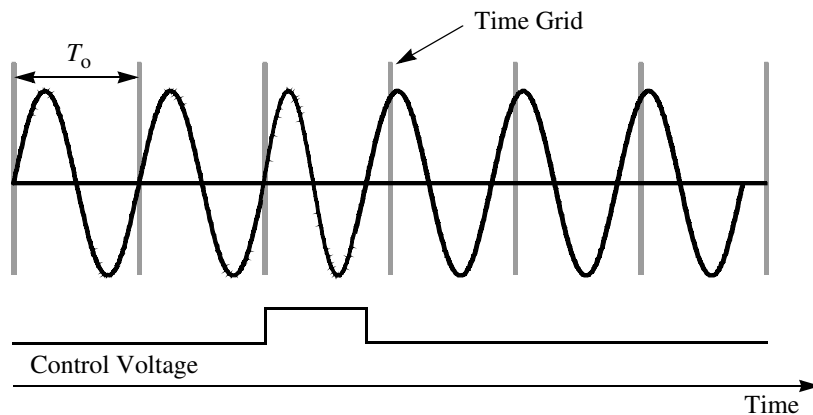
3 Additive White Noise in LC Oscillators

An oscillation is fully characterized by its amplitude and phase. When white noise is added to an oscillation, noise corrupts both the amplitude and phase of oscillation. Assume a noise signal $n(t)$ added to an oscillation $v(t)$. It is customary to model noise as an infinite number of uncorrelated sinusoids separated by 1 Hz each. The sum of the oscillation signal at ω_o and a noise signal at frequency $\omega_o + \omega_n$ is given by:

$$s(t) \ = \ A_o \sin(\omega_o t) + a_n \sin((\omega_o + \omega_n)t + \phi_n), \tag{3}$$

where a_n is the noise amplitude and ϕ_n is a random phase. This equation can re-written as

$$s(t) \ = \ A_o \sin(\omega_o t) + \frac{a_n}{2}\sin((\omega_o + \omega_n)t + \phi_n) + \frac{a_n}{2}\sin((\omega_o - \omega_n)t - \phi_n) \tag{4}$$

$$+ \frac{a_n}{2}\sin((\omega_o + \omega_n)t + \phi_n) - \frac{a_n}{2}\sin((\omega_o - \omega_n)t - \phi_n).$$

The first three terms constitute an amplitude modulated carrier with the modulating tones at ω_n. The last two terms together with the carrier, approximate a narrow band phase modulation signal. This means that a single sideband noise component added to the oscillator modulates both the amplitude and phase of the oscillation. The power of amplitude modulation sidebands is equal to the power of phase modulation sidebands.

In Chapter 2, we will rigorously define phase noise. For now, any noise that modulates the phase of oscillation is phase noise. Any noise that modulates the amplitude is considered amplitude noise and is unimportant in most practical cases, except when it later converts to phase or frequency noise.

4 The Linear Oscillator

4.1 Warning

In this section, we develop a *misleading* analysis of oscillators based on linear system theory. Despite looking reasonable, we will show later why it is not accurate or even correct.

4.2 Linear System Theory Applied to Oscillators

Oscillators are fundamentally nonlinear. In fact oscillator's nonlinearity is the reason for their "stable" amplitude. Yet linear models are often used to describe oscillatory behavior. This is acceptable when oscillation start-up conditions are pursued because oscillation at start-up is a small signal. However, the periodically stable frequency can be far different from the small signal "linear" prediction. Furthermore, a linear oscillator model cannot predict the oscillation amplitude. In fact, the assumption of linearity, leads to an undetermined amplitude. This is because in a linear system, if the input doubles, the output doubles. In an oscillator, this leads to an amplitude that is arbitrary [5].

So what is a linear model good for?

1. It can yields a startup condition for oscillation, and

2. it gives a rough estimate of the frequency of oscillation.

FIGURE 2 *Basic LC oscillator.*
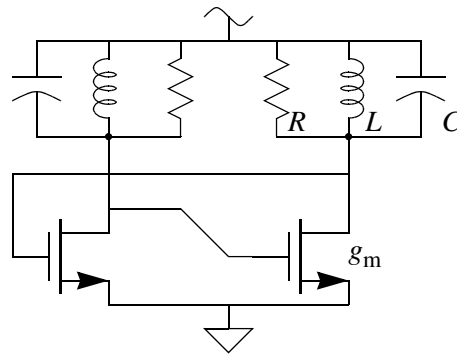
Consider the *LC* oscillator shown in Figure 2. If the oscillator loop is cut at any point, the gain around the loop is given by:

$$G(j\omega) = \left(\frac{-g_{\mathrm{m}}}{\frac{1}{R} + j\omega C - \frac{j}{\omega L}} \right)^2 . \tag{5}$$

For a sustained oscillation, *Barkhausen criterion* mandates that the gain around the loop is exactly unity and the phase shift around the loop is precisely 360 degrees. This leads to the following:

$$g_m R = 1,$$
(6)

$$\omega_o C = \frac{1}{\omega_o L}.$$
(7)

The oscillator shown in Figure 2 can be modeled as a positive feedback system. In Figure 3, the oscillator is constructed using an amplifier and a phase shift network. The amplifier provides no phase shift. The modes of oscillations for this system occur at the natural frequencies of the phase shift network. At these frequencies, the phase shift of this network is a multiple of 360 degrees. If there are multiple frequencies at which Barkhausen criterion is met, then the oscillator can have multiple modes of oscillation. The mode with the highest gain is most likely to prevail but multiple modes of oscillation can coexist.

FIGURE 3 *Feedback amplifier model of the oscillator.*
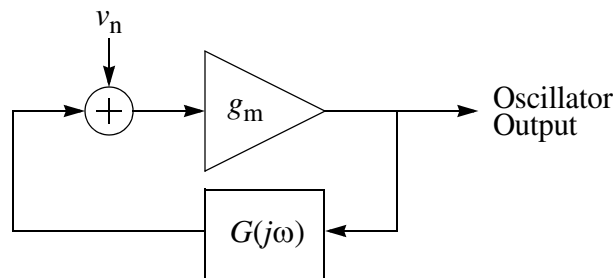
Another way to model an oscillator is a single port model. A lossless *LC* tank is an oscillator with its frequency of oscillation that can be computed from (7). Loss in the tank damps the oscillation with a time constant equal to 1/*RC*. Adding a negative resistance element replenishes any current that flows through the lossy element to sustain oscillation. If the oscillation is to grow then the energy supplied by the negative resistance element must equal the energy lost per cycle. For the oscillator in Figure 2, the two transistors are arranged such that they provide a negative resistance of $-2/g_m$. The differen-

tial resistance of the tank is 2*R*. Therefore, for a sustained oscillation, the negative resistance should be equal to the positive resistance at all times. This leads back to (6). Redrawn in Figure 4, this circuit is known as the *voltage-biased* oscillator.

FIGURE 4 *Basic LC oscillator (redrawn).*
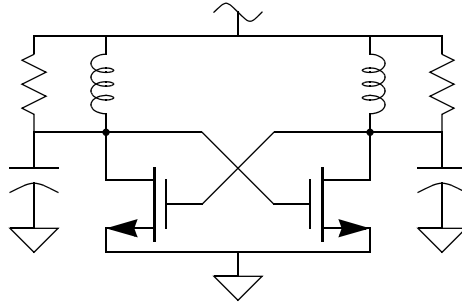
5 Linear Oscillator Noise Analysis

Imagine an oscillator constructed using a parallel *LC* tank and a transconductor in a positive feedback loop. The frequency of oscillation is of course given by (7). Note that the resonant frequency of the *LC* tank is the same as the oscillation frequency because both the resistor and the transconductor do not provide any extra phase shift as they carry no reactive current. The reader can readily prove that if the resistive loss is modeled in series with the inductor rather than it parallel, the oscillation frequency will be different from that given by (7).

Now let's consider the noise. Noise can come from two sources in this system: the resistor and the transconductor. Resistor noise is modeled by a white thermal noise current whose density is given by:

$$\hat{i}_n^2(f) \;=\; \frac{4kT}{R}\,. \tag{8}$$

Noise in the transconductor is also modeled as a white noise current whose density is given by:

$$\hat{i}_n^2(f) = 4kTg_m\gamma, \tag{9}$$

where γ is the noise figure of the transconductor element.

The combined noise can be referred to the input of the transconductor as a white noise voltage given by:

$$\hat{v}_n^2(f) = \frac{4kT\gamma}{g_m} + \frac{4kT}{g_m^2 R}. \tag{10}$$

Substituting from (6), the input referred noise voltage at the transconductor input is given by:

$$\hat{v}_n^2(f) = 4kTR(1+\gamma) = 4kTFR, \tag{11}$$

where $F = 1 + \gamma$ is the noise figure of the entire oscillator.

The tank impedance at a frequency $\delta\omega$ away from the resonance frequency can be approximated by

$$Z(\delta\omega) = \frac{R}{1 + j2Q\dfrac{\delta\omega}{\omega_o}}, \tag{12}$$

where Q is the tank quality factor.

Using basic feedback theory, it is trivial to prove that the closed-loop transfer function from the noise input to the oscillator output is given by:

$$|H(\delta\omega)|^2 = \left(\frac{\omega_o}{2Q\delta\omega}\right)^2. \tag{13}$$

In noise analysis, it is customary to represent noise by a sine wave in a 1 Hz bandwidth. Let's consider a noise component at a frequency $d\omega$ away from the carrier (i.e. oscillation fundamental tone). Noise power at the output of the oscillator can be deduced using (11) and (13),

$$N_{out}(\delta\omega) = 4kTFR\left(\frac{\omega_o}{2Q\delta\omega}\right)^2. \tag{14}$$

As shown earlier, additive noise shows as half amplitude noise and half phase noise. The *noise-to-carrier* ratio is obtained by dividing the output phase noise by the carrier power. We should also consider noise in the lower side band at $\omega_o + \delta\omega$. The single side band noise to carrier ratio is obtained by adding noise power at $+\delta\omega$ and $-\delta\omega$ resulting in the following expression:

$$\mathcal{L}(\delta\omega) = \left(\frac{4kTFR}{P_o}\left(\frac{\omega_o}{2Q\delta\omega}\right)^2 \right)\frac{2}{2} = \frac{kTFR}{P_o}\left(\frac{\omega_o}{Q}\right)^2 \frac{1}{\delta\omega^2}. \tag{15}$$

This is the renowned Leeson's equation [1]. In the original paper it was given as a *heuristic equation without formal proof*. The preceding analysis is not part of the original paper that was based on measurements and observations. Many other researchers and design engineers derived, over the years, proofs similar to the one we derived here and ended up with one version or another of (15).

What's wrong with the above analysis? First, it is linear and time invariant. Therefore, no frequency translations of noise can occur. This means that low frequency noise, such as flicker noise, cannot create phase noise under the assumptions of this model. The only type of noise that can create phase noise in this model is noise originating around the oscillation frequency. Moreover, it has to have an equivalent amount of amplitude noise because it is in essence, additive noise. In any *LC* oscillator, this is not true. As we will show later, some elements contribute pure phase noise and no amplitude noise. Finally, linear analysis cannot predict the amplitude of oscillation. The amplitude limiting mechanism is fundamentally nonlinear and cannot be captured in the context of a linear time-invariant analysis.

6 How Is This Book Different?

In the following chapters we will show in detail why the derivation in Section 4 is wrong. We will show how to use circuit theory to derive an accurate model for phase noise in electrical oscillators. We will describe what we call a mechanistic model that captures the dominant nonlinearities in an oscillator and provides a closed form expression for phase noise. No fudge factors utilized!

Concepts from nonlinear circuit theory are sometimes utilized yet we tried to keep that to the minimum necessary.

By doing so, this book provides deep insight into the operation of oscillators and provides simple procedures for designing high-purity oscillators. We are answering the seldom tackled questions: '*why* does the oscillator behave that way?' and '*how* is an optimal oscillator designed?'

References

[1]   D. B. Leeson, "A simple model of feedback oscillator noise spectrum," *Proceedings of the IEEE*, vol. 54, pp. 329-330, 1966.

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[4]   Q. Huang, "On the exact design of RF oscillators," *Proceedings of the IEEE 1998 Custom Integrated Circuits Conference*, 1998.

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